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Homomorphism and Isomorphism of Rough Group

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Abstract: The chief objective of this study is to show the usefulness of Rough set theory in Group theory. The aim of this paper is to determine the definable and undefinable groups also construct the rough group, upper rough group, homomorphism, and isomorphism of the rough group using rough set theory.

Keywords: Rough Group, Upper Rough Group, Rough Subgroup, Homomorphism and Isomorphism of Upper Rough Group.

1. INTRODUCTION

Rough set theory has been combined with other mathematical theories such as modal logic Boolean algebra, fuzzy sets, semigroup, and random set. Among these research aspects, many papers have been focused on the connection between rough sets and algebraic systems. Biswas and Nanda [1, 2] defined the notion of rough subgroups. Kuroki [4] introduced the notion of a rough idea in a semigroup. In my previous papers{8,9,10], rough graph, rough lattice were defined.In this paper, rough group, the upper rough group are defined and explained about the homomorphism and isomorphism of the upper rough group.

2. ROUGH SET

2.1 Definition [5, 6]

A *rough set* is a formal approximation of a crisp set in terms of a pair of sets which is the lower and upper approximation of the original set. Let U denote the set of objects called universe and let R be an equivalence relation on U. Then (U, R) is called an approximation space. For u, v ∈U & (u, v) ∈ R, u and v belong to the same equivalence class it is denoted by U/R and we say that they are indistinguishable. The relation R is called an indiscernibility relation. Let $[x]_R$ denote an equivalence class of R containing element x then Lower approximation $\underline{R}$(X) and upper approximation $\overline{R}$(X) for a subset X ⊆ U are defined by

$\underline{R}(X) = \{x \in U/[x]_R \subset X\}$

$\overline{R}(X) = \{x \in U/[x]_R \cap X \neq \phi\}$

Thus if an object $x \in \underline{R}$(X) then "x surely belongs to X".If $x \in \overline{R}$(X) then "x possibly belong to X"If $\underline{R}(X)$& $\overline{R}(X)$ are sets then$R(X) = (\underline{R}(X), \overline{R}(X))$ is called a rough set with respect to R.

3. ROUGH GROUP

Definition3.1: Let (U, R) be an approximation space, $*$ be a binary operation defined on U. Let $[x]_R$ denote an equivalence class of R containing element x. Let X⊆ U and the rough set R(X) =($\underline{R}$(X), $\overline{R}$(X)) is called an ***upper rough group*** if it satisfies the following axioms.

A1) for all $x, y \in X, x * y \in \overline{R}$(X)

A2) Association property hold in$\overline{R}$(X)

A3) there exist e ∈ $\overline{R}$(X) such that for all $x \in X. x * e = e * x = x$,e is called the rough identity element of X.

A4) for all $x \in X$ there exist $y \in \overline{R}(X).\ x * y = y * x = e$,$y$ is called the rough inverse element of x in X.

Definition 3.2: Let (U, R) be an approximation space, $*$ be a binary operation defined on U. Let $[x]_R$ denote an equivalence class of R containing element x. Let X⊆ U and R(X) =($\underline{R}$(X), $\overline{R}$(X)) is called *a* ***rough group*** of U, if $\underline{R}$(X) and$\overline{R}$(X) are groups.

Observation 3.3

Let n (R_u(G)) be the number of upper rough group and n(R (G)) be the number of rough group.

(a) The number of upper rough group of (Z_n, $+_n$) under the congruence relation (congruence modulo n/2).If n (n>3) is even then

(i) n(R_u(Z_4)) =2^2+4+2.

(ii) n(R_u(Z_6)) =$2^3+3(2^2)$+6+2.

(iii) n(R_u (Z_8)) =$2^4+4(2^3)+6(2^2)$+8+2.

(iv) n(R_u (Z_{10})) =$2^5+5(2^4)+8(2^3)+8(2^2)+10+2$

(v) n(R_u (Z_{12})) =$2^6+6(2^5)+10(2^4)+10(2^3)+10(2^2)+12+2$

In general

$$n(R_u(Z_n))=2^{\frac{n}{2}}+\frac{n}{2}\left(2^{\frac{n}{2}-1}\right)+(n-2)\left(2^{\frac{n}{2}-2}\right)+\cdots+(n-2)(2^2)+n+2$$

(b) The number of upper rough group of (Z_n,$+_n$) under the congruence relation(congruence modulo$\frac{n-1}{2}$) If n(n>3) is odd then

(i) n(R_u (Z_5)) =6(2) +5+3.

(ii) n(R_u (Z_7)) =6(2^2)+12(2)+2^2+7+3

(iii) n(R_u (Z_9)) =6(2^3)+18(2^2)+18(2)+2^3+2^2+9+3

(iv) n(R_u (Z_{11}))=6(2^4)+ 24(2^3) +24(2^2)+24(2)+$2^4+2^3+2^2$+11+3

(v) n(R_u (Z_{13})) =$6(2^5)+30(2^4)+30(2^3)+30(2^2)+30(2)+2^5+2^4+2^3+2^2+13+3$

In general $n(R_u(Z_n))=6(2^{\frac{n-3}{2}})+6\left(\frac{n-3}{2}\right)\left(2^{\frac{n-5}{2}}\right)+\cdots+6\left(\frac{n-3}{2}\right)\left(2^{\frac{n-(n-2)}{2}}\right)+2^{\frac{n-3}{2}}+\cdots+\left(\left(2^{\frac{n-(n-4)}{2}}\right)\right)+n+3$

Observation 3.4

(a) The number of rough group of (Z_n, $+_n$) under the congruence relation (congruence modulo n/2) If n(n>3) is even then

(i) n(R(Z_4))=.$\binom{2}{1}$

(ii) n(R(Z_6))=$\binom{2}{1}^2$

(iii) n(R(Z_8)) =$\binom{2}{1}+\binom{2}{1}^3$

(iv) n(R(Z_{10})) =$(\binom{2}{1})^4$

(v) n(R(Z_{12})) =$\binom{2}{1}+\binom{2}{1}^5$

In general $n(R(Z_n))=\begin{cases}\binom{2}{1}+\binom{2}{1}^{\frac{n}{2}-1} & if\ \frac{n}{2}\ is\ even\\ \binom{2}{1}^{\frac{n}{2}-1} & if\ \frac{n}{2}\ is\ odd\end{cases}$

(b) The number of rough group of (Z_n, $+_n$) under the congruence relation (congruence modulo$\frac{n-1}{2}$) If n(n>3) is odd then

n(R (Z_n) =0

Theorem 3.5: Let (U, R) be an approximation space, $*$ be a binary operation defined on U. And U be a group with respect to *. If $\overline{R}$(X) is a sub group then X is an upper rough subgroup.

Proof

(i) Since $\overline{R}$(X) is a subgroup and X⊆ $\overline{R}$(X) for all $x,y\in X, x*y\in\overline{R}$(X). Association property hold's in$\overline{R}$(X). And there exist e∈ $\overline{R}$(X) such that for all$x\in X. x*e=e*x=x$, e is called the rough identity element of X. for all$x\in X$ there exist $y\in\overline{R}$(X), $x*y=y*x=e$,y is called the rough inverse element of x in X. Thus X is an upper rough subgroup of U.

Remark 3.6

The converse of the above theorem need not be true.

for example.

Let U be the set of all permutation of S_4 and $*$be the multiplication operation of permutation. A classification of U is $U/R=\{E_1,E_2,E_3,E_4\}$, where

E_1 = {(1), (12), (13), (14), (23), (24), (34)},

E_2 = {(123), (132), (124), (142), (134), (143), (234), (243)},

E_3 = {(1234), (1243), (1324), (1342), (1423), (1432)},

E_4 = {(12)(34), (13)(24), (14)(23)},

Let X = {(12), (123), (132)}, then

$\overline{R}$(X) = $E_1\cup E_2$

(i) $\forall x, y \in X, x * y \in \overline{R}(X)$
(ii) $(12) * (12) = (1) \in \overline{R}(X)$;
(iii) *Association property holds in* $\overline{R}(X)$;
(iv) $(12)^{-1} = (12) \in \overline{R}(X), (123)^{-1} = (132) \in \overline{R}(X)$,
$(132)^{-1} = (123) \in \overline{R}(X)$.

Thus we have X is an upper rough subgroup.
Here $(123),(234) \in \overline{R}(X)$, but $(123) * (234) = (13)(24) \notin \overline{R}(X)$.Hence $\overline{R}(X)$ is not a sub group.

4. HOMOMORPHISM AND ISOMORPHISM OF ROUGH GROUP

Let $(U, R),(U'R')$ be two approximation spaces. Let U and U' be the groups with respect to the binary operations $*, \overline{*}$ respectively. Let $X \subseteq U, Y \subseteq U'$ such that $\overline{R}(X), \overline{R}(X')$ be two sub groups of U, U' respectively.If $f: \overline{R}(X) \to \overline{R}(X')$ is a homomorphism, then we say that $f: X \to X'$ is an *upper rough homomorphism* such that for all $x, y \in \overline{R}(X)$, $f(x * y) = f(x) \overline{*} f(y)$.

Theorem 4.1: Let $(U, R),(U', R')$ be two approximation spaces, Let U and U' be the groups with respect to the binary operations $*, \overline{*}$ respectively. Let $X \subseteq U, Y \subseteq U'$ such that $\overline{R}(X), \overline{R}(X')$ are two sub group of U, U' respectively. If f: $\overline{R}(X) \to \overline{R}(X')$ is a homomorphism, $f: X \to X'$ is an upper rough homomorphism, then the following properties hold.

(i) $f(e) = e'$

(ii) $f(a^{-1}) = [f(a)]^{-1}$

Proof

(i) Let $a \in X$ then $a \in \overline{R}(X)$ there exist $e \in \overline{R}(X)$ such that $a * e = e * a = e$ then $f(a * e) = f(e * a) = f(a)$
$f(a) \overline{*} f(e) = f(e) \overline{*} f(a) = f(a)$

Hence $f(e) = e'$, $e' \in \overline{R}(X')$

(ii) Let $a \in X$ then $a^{-1} \in \overline{R}(X)$ such that $a * a^{-1} = a^{-1} * a = e$
$f(a * a^{-1}) = f(a^{-1} * a) = f(e)$
$f(a) \overline{*} f(a^{-1}) = f(a^{-1}) \overline{*} f(a) = f(e) = e'$
Hence $f(a^{-1}) = [f(a)]^{-1}$

Definition 4.2: Let $(U, R),(U'R')$ be two approximation spaces. Let U and U' be the groups with respect to the binary operations $*, \overline{*}$ respectively. $X \subseteq U, X' \subseteq U'$ such that $\overline{R}(X), \overline{R}(X')$ are sub groups of U, U' respectively. If $f: \overline{R}(X) \to \overline{R}(X')$ is a isomorphism, then we say that $f: X \to X'$ is an *upper rough Isomorphism.*

Theorem 4.3: Let $(U, R),(U', R')$ be two approximations spaces, where U and U' are groups with respect to the binary operations $*, \overline{*}$ respectively. Let $X \subseteq U, X' \subseteq U'$ such that $\overline{R}(X), \overline{R}(X')$ are sub groups of U, U' respectively. If $f: \overline{R}(X) \to \overline{R}(X')$ is an isomorphism, then

(i) $f\left(\overline{R}(X)\right) = \overline{R}(f(X))$

(ii) $f^{-1}\left(\overline{R}(X)\right) = \overline{R}(f^{-1}(X))$

Proof

(i) $f\left(\overline{R}(X)\right) = \overline{R}(f(X))$

Let $x \in f\left(\overline{R}(X)\right)$

$\Rightarrow f^{-1}(x) \in \overline{R}(X)$,
$\Rightarrow [f^{-1}(x)]_R \cap X \neq \phi$
$\Rightarrow f([f^{-1}(x)]_R \cap X) \neq \phi$
$\Rightarrow f([f^{-1}(x)]_R) \cap f(X) \neq \phi$
$\Rightarrow [x]_R \cap f(x) \neq \phi$ [since f is an isomorphism]
$\Rightarrow x \in \overline{R}(f(X))$
$f\left(\overline{R}(X)\right) \subseteq \overline{R}(f(X))$

Let $x \in \overline{R}(f(X))$
$\Rightarrow x \in [f(x)]_R \cap f(X)$
$\Rightarrow x \in f(X)$
$\Rightarrow x \in f\left(\overline{R}(X)\right)$[since $X \subseteq \overline{R}(X)] \Rightarrow f(X) \subseteq f(\overline{R}(X))$]
$\overline{R}(f(X)) \subseteq f\left(\overline{R}(X)\right)$
Hence $f\left(\overline{R}(X)\right) = \overline{R}(f(X))$

(ii)$f^{-1}\left(\overline{R}(X)\right) = \overline{R}(f^{-1}(X))$
Let $x \in f^{-1}\left(\overline{R}(X)\right)$
$\Longrightarrow f(x) \in \overline{R}(X),$
$\Longrightarrow [f(x)]_R \cap X \neq \phi$
$\Longrightarrow f^{-1}([f(x)]_R \cap X) \neq \phi$
$\Longrightarrow f^{-1}([f(x)]_R) \cap f^{-1}(X) \neq \phi$
$\Longrightarrow [x]_R \cap f^{-1}(x) \neq \phi$ [since f is an isomorphism]
$\Longrightarrow x \in \overline{R}(f^{-1}(X))$
$f^{-1}\left(\overline{R}(X)\right) \subseteq \overline{R}(f^{-1}(X))$
Let $x \in \overline{R}(f^{-1}(X))$
$\Longrightarrow x \in [f^{-1}(x)]_R \cap f^{-1}(X)$
$\Longrightarrow x \in f^{-1}(X)$
$\Longrightarrow x \in f^{-1}\left(\overline{R}(X)\right)$[Since$X \subseteq \overline{R}(X)] \Longrightarrow f^{-1}(X) \subseteq f^{-1}(\overline{R}(X))$]
$\overline{R}(f^{-1}(X)) \subseteq f^{-1}\left(\overline{R}(X)\right)$
Hence $f^{-1}\left(\overline{R}(X)\right) = \overline{R}(f^{-1}(X))$

Theorem 4.4: Let(U,R),(U',R') be two approximation spaces, Let $U$and U'be the groups with respect to the binary operations $*$,$\overline{*}$ respectively. Let $X \subseteq U, X' \subseteq U'$ such that $\overline{R}(X)$,$\overline{R}(X')$ be two sub group of U, U'respectively. If f: $\overline{R}(X) \to \overline{R}(X')$ is an isomorphism, f : X $\to X'$ is an upper rough isomorphism, then the following properties hold.

(i) fIf X_1 is an upper rough sub group of X then$f(X_1)$ is an upper rough subgroup ofX'

(ii) If X_1 is an upper rough normal in X then$f(X_1)$ is an upper rough normal in $f(X)$

(iii) If ${X_1}'$ is an upper rough sub group of X' then$f^{-1}({X_1}')$ is an upper rough subgroup of$X$

(iv) If ${X_1}'$ is an upper rough normal inX' then$f^{-1}({X_1}')$ is an upper rough normal in$X$

Proof

(i) If X_1 is an upper rough sub group of X
Since X_1 is non empty,$f(X_1)$ is also non empty.
Now let $x,y \in f(X_1)$ then$x,y \in f\left(\overline{R}(X_1)\right)$,$x = f(a), y = f(b)$ where $a,b \in \overline{R}(X_1)$
Therefore $x \,\overline{*}\, y^{-1} = f(a) \,\overline{*}\, [f(b)]^{-1} = f(a) \,\overline{*}\, f(b^{-1}) = f(a * b^{-1})$
Since X_1 is an upper rough sub group of X, $a * b^{-1} \in \overline{R}(X_1)$
Therefore $x \,\overline{*}\, y^{-1} = f(a * b^{-1}) \in f(\overline{R}(X_1))$
Therefore $f(\overline{R}(X_1))$ is a subgroup of X'
By Theorem 4.3 $f\left(\overline{R}(X)\right) = \overline{R}(f(X))$
$\overline{R}(f(X_1))$ is a subgroup of X'
By Theorem 3.5
$f(X_1)$ is an upper rough subgroup of X'

(ii) Let X_1 is an upper rough normal in X
Let$x \in f(X_1)$ and$y \in f(X)$
claim$y * x * y^{-1} \in \overline{R}(f(X_1))$
Now, $x = f(a), y = f(b)$ where $a \in X_1$&$b \in X$
Since X_1 is an upper rough normal in X,$\overline{R}(X_1)$is normal in,
$b * a * b^{-1} \in \overline{R}(X_1)$
Therefore $f(b * a * b^{-1}) \in f(\overline{R}(X_1))$
$f(b) \,\overline{*}\, f(a) \,\overline{*}\, f(b^{-1}) \in f(\overline{R}(X_1))$

Therefore$y \,\overline{*}\, x \,\overline{*}\, y^{-1} \in f\left(\overline{R}(X_1)\right)$
$f\left(\overline{R}(X_1)\right)$is normal in $f(X)$
By Theorem 4.3 $\overline{R}(f(X_1))$ is normal in $f(X)$
$\Longrightarrow f(X_1)$is an upper normal in $f(X)$

(iii) Let ${X_1}'$ be an upper rough sub group of X'
Since $(e) = e' \in \overline{R}({X_1}')$, $e \in f^{-1}(\overline{R}({X_1}'))$
hence$f^{-1}(\overline{R}({X_1}')) \neq \varphi$
Let $a,b \in f^{-1}\left(\overline{R}({X_1}')\right)$ then $f(a), f(b) \in \overline{R}({X_1}')$
$\Rightarrow f(a) \,\overline{*}\, [f(b)]^{-1} \in \overline{R}({X_1}')$

$\Rightarrow f(a * b^{-1}) \in \overline{R}(X_1')$
$\Rightarrow a * b^{-1} \in f^{-1}(\overline{R}(X_1'))$
$f^{-1}(\overline{R}(X_1'))$is a subgroup of X
$\Longrightarrow \overline{R}(f^{-1}(X_1')$is sub group of X.
$\Longrightarrow f^{-1}(X_1')$is an upper rough subgroup of X
Hence$f^{-1}(X_1'')$ is an upper rough subgroup of$X$

(iv) Let X_1' is an upper rough normal inX'
Let $\in f^{-1}(X_1')$ and$a \in X$
Then $f(x) \in X_1'$ and$f(a) \in f(X)$
To prove$a * x * a^{-1} \in f^{-1}(X_1')$
Since X_1' is an upper rough normal in X', $\overline{R}(X_1')$is normal in X'

$$\Rightarrow f(a) \mathbin{\overline{*}} f(x) \mathbin{\overline{*}} [f(a)]^{-1} \in \overline{R}(X_1')$$
$$\Rightarrow f(a * x * a^{-1}) \in (\overline{R}(X_1))$$
$$\Rightarrow a * x * a^{-1} \in f^{-1}(\overline{R}(X_1'))$$

$f^{-1}(\overline{R}(X_1'))$is normal in X
$\Longrightarrow \overline{R}(f^{-1}(X_1'))$is normal in X [by theorem 4.3]
$f^{-1}(X_1')$ is an upper rough normal in$X$.

Theorem 4.5: An upper rough isomorphism is an equivalence relation
Proof

Let $\overline{R}(X)$,$\overline{R}(X')$ be two sub groups of U and U' respectively, then by theorem 3.5, X and X' be two upper rough sub groups of U and U' respectively.

For any group $\overline{R}(X)$, $i_x = \overline{R}(X) \rightarrow \overline{R}(X')$ is an isomorphism clearly the identity map $i_x = X \rightarrow X'$is an upper rough isomorphism,
hence$X \cong_r X$

Therefore the relation is reflexive.

Let $\overline{R}(X)$,$\overline{R}(X')$ be two sub group of U, U'respectively,then X ,X'be two upper rough sub group of U, U'respectively

Now let $X \cong_r X'$and let $f : \overline{R}($ X$) \rightarrow \overline{R}(X')$ be an isomorphism,
$\Longrightarrow f : \overline{R}($ X$) \rightarrow \overline{R}(X')$is a bijection.
$\Longrightarrow f^{-1}: \overline{R}(X') \rightarrow \overline{R}(X)$is also a bijection
Let $x', y' \in \overline{R}(X')$
Since f is onto there exists $x \in X, y \in X$
such that $x' = f(x)$ and $y' = f(y)$
$\Longrightarrow x = f^{-1}(x') and\ y = f^{-1}(y')$
Therefore $f(xy) = f(x)f(y) = x'y'$
Therefore $f^{-1}(x'y') = xy = f^{-1}(x')f^{-1}(y')$
Therefore$f^{-1}: \overline{R}(X') \rightarrow \overline{R}(X)$ is a homomorphism
Therefore$f^{-1}: \overline{R}(X') \rightarrow \overline{R}(X)$ is an isomorphism
Hence $f^{-1}: X' \rightarrow X$is an upper rough isomorphism Hence $X' \cong_R X$ and
The relation sis symmetric
Now let $X \cong_r X'$and$X' \cong_r X''$and let $f: \overline{R}(X) \rightarrow \overline{R}(X')$ be an isomorphism, $h: \overline{R}(X') \rightarrow \overline{R}(X'')$ is an isomorphism. $hof: \overline{R}(X) \rightarrow \overline{R}(X')$is also rough bijection
Now let $x, y \in X$ then

$$(hof)(xy) = h[f(xy)]$$
$$= h[f(x)(y)]$$
$$= h(f(x))h(f(y))$$
$$= (hof)(x)(hof)(y)$$

$hof: \overline{R}(X) \rightarrow \overline{R}(X'')$is an isomorphism

Hence $hof\ X \rightarrow X''$is an upper rough isomorphism Thus$X \cong_r X''$ and hence the relation is transitive.
Therefore an upper rough isomorphism is an equivalence relation.

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